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LETTER TO THE EDITOR

Mapping of shape invariant potentials under point canonical transformations

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Abstract. We give explicit point canonical transformations which map twelve types of shape invariant potentials (which are known to be exactly solvable) into two potential classes. The eigenfunctions in these two classes are given by hypergeometric and confluent hypergeometric functions respectively.

The application of supersymmetry to quantum mechanics [1] has revived fresh interest in the problem of obtaining algebraic solutions of exactly solvable non-relativistic potentials. In an interesting paper, Gendenshtein [2] showed that whenever a parametric relation, the so-called 'shape invariance' condition, is satisfied by two supersymmetric partner potentials, the bound state spectra and eigenfunctions can be readily determined by purely algebraic means using factorizability of the Hamiltonians. This generalization is in many respects equivalent to the earlier work of Schrödinger [3] and Infeld and Hull [4]. Using the concept of shape invariance, Dutt *et al* [5] have explicitly worked out the bound state spectra for eleven types of shape invariant potentials. Subsequently, using the operator formalism, Dabrowska *et al* [6] have shown an elegant way of writing the eigenfunctions for all these problems. Recently, Barclay and Maxwell [7] have discussed one more type of shape invariant potential. (This corresponds to the superpotential $W = A\tan(\alpha x) + B/A$, and is the trigonometric version of the Rosen–Morse potential.)

There also exist other solvable potentials for which the factorization procedure is not applicable, since they are not shape invariant [8, 9]. It has been shown by Cooper *et al* [9] that many such potentials (for example, the Natanzon potentials [10]), can be generated by applying an operator transformation (f-transformation) to shape invariant potentials. This procedure neither preserves shape invariance nor, in general, transforms a potential into its supersymmetric partner potential. In fact, the general method of operator f-transformations yields new solvable potentials and does not depend on supersymmetry. Alternatively, the techniques of supersymmetric quantum mechanics can be used to generate multi-parameter families of solvable potentials which are strictly isospectral to any given shape invariant potential [11]. The number of solvable families can be yet further enlarged by using the Abraham–Moses and Pursey procedures for deleting and inserting bound states [12] instead of the customary Darboux procedure used in supersymmetric quantum mechanics.

At this stage, it is quite natural to ask whether it is possible to interrelate the twelve known types of shape invariant potentials among themselves via transformations analogous to the f-transformation. We find that this is indeed the case; the known types of shape invariant potentials can be grouped into two classes in the sense that the potentials in any class can all be mapped to a single potential in that class through point canonical transformations (PCT) [13]. PCT have been studied in the path integral approach to quantum mechanical problems [14, 15]. Pak and Sokmen [16] and Inomata [17] suggested that PCT together with a path dependent time transformation (local time transformation) could reduce a few solvable potential kernels to the kernels of either the harmonic oscillator or Scarf potentials [15, 18]. However, in path integral calculations, the mathematical manoeuvring of steps becomes so complicated due to the combined transformations of space and time variables that the mapping of all shape invariant potentials has not yet been done.

In this letter, we show that a much simpler approach consists of mapping through canonical transformation of coordinates which interrelate the Hilbert spaces of various shape invariant potentials. A similar suggestion has been made recently by Junker [19]. The general method of transformation of the time-independent Schrödinger equation into a hypergeometric equation goes back to Manning [20]. The method was further studied by other authors [21]. We re-establish the known result that the Coulomb and Morse potentials can be mapped into the three-dimensional harmonic oscillator. These types of potentials form one class. For these class I potentials, the eigenfunctions correspond to confluent hypergeometric functions which can be written as Laguerre polynomials. Furthermore, using PCT, we show that potentials such as the Rosen–Morse (hyperbolic and trigonometric), Eckart, Poschl–Teller (I and II), etc. can be mapped into the generalized Scarf potential, and they form a second class. For these class II potentials, the eigenfunctions correspond to hypergeometric functions which can be written as associated Legendre functions.

After presenting the general formulation of PCT as applied to the Schrödinger equation, we illustrate the procedure with a simple example. We show the steps necessary to connect the energy eigenvalues and eigenfunctions of the hyperbolic Rosen–Morse potential with those of the generalized Scarf potential, which corresponds to the superpotential $W=-A\cot(ax)+B\,\mathrm{cosec}(ax), 0<ax<\pi, A>B$. For all other types of shape invariant potentials, we cite the appropriate transformations of coordinates and the energy eigenvalues and eigenfunctions in tables 1 and 2. From the tables, one can easily find the sequence of transformations necessary to map any potential of a given class to another one belonging to the same class. In this respect, our work is related to the group theoretical classification of solvable potentials [22, 23].

First we give the general PCT which transforms the time-independent Schrödinger equation for a given shape invariant potential $V(\alpha_i;x)$.

$$\left[\frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(\alpha_i;x)-E(\alpha_i)\right]\psi(\alpha_i;x)=0 \tag{1}$$

to a corresponding one

$$\left[\frac{-\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\tilde{V}(\tilde{\alpha}_i;z)-\tilde{E}(\tilde{\alpha}_i)\right]\tilde{\psi}(\tilde{\alpha}_i;z)=0 \tag{2}$$

for which $\tilde{\psi}_n(\tilde{\alpha}_i;z)$ and $\tilde{E}_n(\tilde{\alpha}_i)$ are assumed to be known for the shape invariant potential $\tilde{V}(\tilde{\alpha}_i;z)$ for each state labelled by the quantum number $n=0,1,2,\ldots$. Here $\{\alpha_i\}$ and $\{\tilde{\alpha}_i\}$ represent sets of parameters of the original (old) and transformed (new) potentials respectively.

Table 1. Interrelations of energy eigenvalues and eigenfunctions obtained by mapping via PCT among shape invariant potentials of class I, i.e. those whose eigenfunctions correspond to confluent hypergeometric functions.

Name of potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among parameters	Eigenvalues $E_n^{(-)}$	Variable y	Radial part of wavefunction ψ_n
Harmonic oscillator	$\sqrt{\frac{m}{2}}\bar{\omega}z-\frac{(\bar{l}+1)\hbar}{\sqrt{2m}z}$ $(0\leqslant z<\infty)$	$\frac{1}{2}m\bar{\omega}^2z^2+\frac{\bar{l}(\bar{l}+1)\hbar^2}{2mz^2}$ $-(\bar{l}+\frac{3}{2}\hbar\bar{\omega})$	—	—	$2n\hbar\bar{\omega}$	$y=m\bar{\omega}z^2$	$\exp(-\frac{1}{2}y)y^{(\bar{l}+1)/2}$ $\times L_n^{\bar{l}+\frac{1}{2}}(y)$
Coulomb	$\sqrt{\frac{m}{2}}\frac{e^2}{(l+1)\hbar}$ $-\frac{(l+1)\hbar}{\sqrt{2m}x}$ $(0\leqslant x<\infty)$	$-\frac{e^2}{x}+\frac{l(l+1)\hbar^2}{2mx^2}$ $+\frac{me^4}{2(l+1)^2\hbar^2}$	z^2	$\bar{\omega}=\frac{2e^2}{\hbar(n+l+1)}$ $\bar{l}=2l+\frac{1}{2}$	$\frac{me^4}{2\hbar^2}\left(\frac{1}{(l+1)^2}-\frac{1}{(n+l+1)^2}\right)$	$y=\frac{2me^2x}{\hbar(n+l+1)}$	$y^{l+1}\exp(-\frac{1}{2}y)$ $\times L_n^{2l+1}(y)$
Morse	$A-B\,e^{-\alpha x}$ $(-\infty<x<\infty)$	$A^2+B^2\,e^{-2\alpha x}$ $-2B\left(A+\frac{\alpha\hbar}{2\sqrt{2m}}\right)e^{-\alpha x}$	$-\frac{2}{\alpha}\ln z$	$\bar{\omega}=\frac{2\sqrt{2}B}{\alpha\sqrt{m}}$ $\bar{l}=\left(\frac{2\sqrt{2m}A}{\hbar\alpha}-2n-\frac{1}{2}\right)$	$A^2-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y=\frac{2\sqrt{2m}B}{\alpha}e^{-\alpha x}$	$y^{(\sqrt{2m}A/\hbar\alpha-n)}\exp(-\frac{1}{2}y)$ $\times L_n^{(2\sqrt{2m}A/\hbar\alpha-2n)}(y)$

Table 2. Interrelations of energy eigenvalues and eigenfunctions obtained by mapping via PCT among shape invariant potentials of class II, i.e. those whose eigenfunctions correspond to hypergeometric functions.

Name of potential	Superpotential W	Potential V	Transformation $f(z)$	Relation among parameters	Eigenvalues E_n	Variable y	Radial part of wavefunction ψ_n
Scarf	$-\tilde{A}\cot az$ $+\tilde{B}\operatorname{cosec} az$ $(0\leqslant az\leqslant\pi;$ $\tilde{A}>\tilde{B})$	$-\tilde{A}^2+\left(\tilde{A}^2+\tilde{B}^2-\frac{\tilde{A}a\hbar}{\sqrt{2m}}\right)\operatorname{cosec}^2 az$ $-\tilde{B}\left(2\tilde{A}-\frac{a\hbar}{\sqrt{2m}}\right)\cot az$ $\times\operatorname{cosec} az$	—	—	$\left(\tilde{A}+\frac{na\hbar}{\sqrt{2m}}\right)^2-\tilde{A}^2$	$y=\cos az$ $s=\frac{\sqrt{2m}}{\hbar}\frac{\tilde{A}}{a}$ $\lambda=\frac{\sqrt{2m}}{\hbar}\frac{\tilde{B}}{a}$	$(1-y)^{(s-\lambda)/2}$ $\times(1+y)^{(s+\lambda)/2}$ $\times P_n^{(s-\lambda-\frac{1}{2},s+\lambda-\frac{1}{2})}(y)$
Scarf	$A\tanh ax$ $+B\operatorname{sech} ax$ $(-\infty<x<\infty)$	$A^2+\left(B^2-A^2-\frac{Aa\hbar}{\sqrt{2m}}\right)\operatorname{sech}^2 ax$ $+B\left(2A+\frac{a\hbar}{\sqrt{2m}}\right)\operatorname{sech} ax$ $\times\tanh ax$	$\frac{\sinh^{-1}(-\mathrm{i}\cos az)}{a}$	$\tilde{A}=-A$ $\tilde{B}=\mathrm{i}B$	$A^2-\left(A-\frac{na\hbar}{\sqrt{2m}}\right)^2$	$y=\sinh ax$ $s=\frac{\sqrt{2m}A}{\hbar a}$ $\lambda=\frac{\sqrt{2m}B}{\hbar a}$	$(1+y^2)^{-s/2}$ $\times\exp(-\lambda\tan^{-1}y)$ $\times P_n^{(-s-\mathrm{i}\lambda-\frac{1}{2},-s+\mathrm{i}\lambda-\frac{1}{2})}(\mathrm{i}y)$
Rosen–Morse I	$A\tan ax+\frac{B}{A}$ $(-\infty<x<\infty)$	$-A^2+\frac{B^2}{A^2}$ $+2B\tan ax$ $+A\left(A-\frac{a\hbar}{\sqrt{2m}}\right)\sec^2 ax$	$\frac{\cos^{-1}(\operatorname{cosec} az)}{a}$	$\tilde{A}=-A-\left(n-\frac{1}{2}\right)\frac{\hbar a}{\sqrt{2m}}$ $\tilde{B}=-\frac{\mathrm{i}B}{\left(A+\frac{na\hbar}{\sqrt{2m}}\right)}$	$\left(A+\frac{na\hbar}{\sqrt{2m}}\right)^2-A^2$ $+\frac{B^2}{A^2}-\frac{B^2}{\left(A+\frac{na\hbar}{\sqrt{2m}}\right)^2}$	$y=\tan ax$ $s=\frac{\sqrt{2m}A}{\hbar a}$ $\lambda=\frac{\sqrt{2m}B}{\hbar a}$ $a=\frac{\sqrt{2m}\lambda}{\hbar a(s+n)}$	$(1+y^2)^{-(s+n)/2}$ $\times\exp(-a\tan^{-1}y)$ $\times P_n^{(-s-n+\mathrm{i}a,s-n-\mathrm{i}a)}(-\mathrm{i}y)$
Rosen–Morse II	$A\tanh ax+\frac{B}{A}$ $(-\infty<x<\infty)$	$A^2+\frac{B^2}{A^2}$ $+2B\tanh ax$ $-A\left(A+\frac{a\hbar}{\sqrt{2m}}\right)\operatorname{sech}^2 ax$	$\frac{\tanh^{-1}(\cos az)}{a}$	$\tilde{A}=A-\left(n-\frac{1}{2}\right)\frac{\hbar a}{\sqrt{2m}}$ $\tilde{B}=-\frac{B}{\left(A-\frac{na\hbar}{\sqrt{2m}}\right)}$	$A^2-\left(A-\frac{na\hbar}{\sqrt{2m}}\right)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{\left(A-\frac{na\hbar}{\sqrt{2m}}\right)^2}$	$y=\tanh ax$ $s=\frac{\sqrt{2m}A}{\hbar a}$ $\lambda=\frac{\sqrt{2m}B}{\hbar a}$ $a=\frac{\sqrt{2m}\lambda}{\hbar a(s-n)}$	$(1-y)^{(s-n+a)/2}$ $\times(1+y)^{(s-n-a)/2}$ $\times P_n^{(s-n+a,s-n-a)}(y)$

Table 2. (continued)

Rosen–Morse II	$A\coth\alpha x$ $-B\,\mathrm{cosech}\,\alpha x$ $(0\leqslant x<\infty;$ $A<B)$	$A^2+\left(B^2+A^2+\frac{A\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosech}^2\,\alpha x$ $-B\left(2A+\frac{\alpha\hbar}{\sqrt{2m}}\right)\coth\alpha x$ $\times\,\mathrm{cosech}\,\alpha x$	$\frac{\cosh^{-1}(\cos\alpha z)}{\alpha}$	$\tilde{A}=-A$ $\tilde{B}=-B$	$A^2-\left(A-\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y=\cosh\alpha x$ $s=\frac{\sqrt{2m}A}{\hbar\alpha}$ $\lambda=\frac{\sqrt{2m}B}{\hbar\alpha}$	$(y-1)^{(\lambda-s)/2}$ $\times(1+y)^{-(\lambda+s)/2}$ $\times P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(y)$
Eckart	$-A\coth\alpha x+\frac{B}{A}$ $(0\leqslant x<\infty;$ $B>A^2)$	$A^2+\frac{B^2}{A^2}$ $-2B\coth\alpha x$ $+A\left(A-\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosech}^3\,\alpha x$	$\frac{\coth^{-1}(\cos\alpha z)}{\alpha}$	$\tilde{A}=-A-\left(n-\frac{1}{2}\right)\frac{\hbar\alpha}{\sqrt{2m}}$ $\tilde{B}=-\frac{B}{\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)}$	$A^2-\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2$ $+\frac{B^2}{A^2}-\frac{B^2}{\left(A+\frac{n\alpha\hbar}{\sqrt{2m}}\right)^2}$	$y=\coth\alpha x$ $s=\frac{\sqrt{2m}A}{\hbar\alpha}$ $\lambda=\frac{\sqrt{2m}B}{\hbar\alpha}$ $a=\frac{\sqrt{2m}\lambda}{\hbar(s-n)}$	$(y-1)^{-(s+n-a)/2}$ $\times(y+1)^{-(s+n+a)/2}$ $\times P_n^{(-s+a-n,-s-a-n)}(y)$
Poschl–Teller I	$A\tan\alpha x$ $-B\cot\alpha x$ $\left(0\leqslant\alpha x\leqslant\frac{\pi}{2}\right)$	$-(A+B)^2$ $+A\left(A-\frac{\alpha\hbar}{\sqrt{2m}}\right)\sec^2\alpha x$ $+B\left(B-\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosec}^2\,\alpha x$	$z/2$	$\tilde{A}=\left(\frac{A+B}{2}\right)$ $\tilde{B}=\left(\frac{A-B}{2}\right)$	$\left(A+B+\frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$ $-(A+B)^2$	$y=1-2\sin^2\alpha x$ $s=\frac{A}{\alpha}$ $\lambda=\frac{B}{\alpha}$	$(1-y)^{\lambda/2}(1+y)^{s/2}$ $\times P_n^{(\lambda-\frac{1}{2},s-\frac{1}{2})}(y)$
Poschl–Teller II	$A\tanh\alpha x$ $-B\coth\alpha x$ $(0\leqslant x<\infty;$ $B<A)$	$(A-B)^2$ $-A\left(A+\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{sech}^2\,\alpha x$ $+B\left(B-\frac{\alpha\hbar}{\sqrt{2m}}\right)\mathrm{cosech}^2\,\alpha x$	$\frac{\sinh^{-1}\left(\mathrm{i}\sin\frac{\alpha z}{2}\right)}{\alpha}$	$\tilde{A}=\left(\frac{B-A}{2}\right)$ $\tilde{B}=-\left(\frac{A+B}{2}\right)$	$(A-B)^2$ $-\left(A-B-\frac{2n\alpha\hbar}{\sqrt{2m}}\right)^2$	$y=1+2\sinh^2\alpha x$ $s=\frac{A}{\alpha}$ $\lambda=\frac{B}{\alpha}$	$(1-y)^{\lambda/2}(1+y)^{-s/2}$ $\times P_n^{(\lambda-\frac{1}{2},-s-\frac{1}{2})}(y)$

Invoking a transformation of both the independent and dependent variables of the form

$$x = f(z) \qquad \psi(\alpha_i; x) = v(z)\tilde{\psi}(\tilde{\alpha}_i; z) \tag{3}$$

equation (1) becomes

$$\frac{-\hbar^2}{2m}\frac{d^2\tilde{\psi}}{dz^2} - \frac{\hbar^2}{m}\frac{d\tilde{\psi}}{dz}\left\{\frac{v'}{v} - \frac{f''}{2f'}\right\} + \left[f'^2\{V(\alpha_i; f(z)) - E(\alpha_i)\} + \frac{\hbar^2}{2m}\left\{\frac{f''v'}{f'v} - \frac{v''}{v}\right\}\right]\tilde{\psi} = 0 \tag{4}$$

in which the prime denotes differentiation with respect to the variable z. To remove the first derivative term in (4) one requires

$$v(z) = C\sqrt{f'(z)} \tag{5}$$

where C is a constant of integration. For a known transformation function f, one then finds the wavefunction of the original problem in terms of the known eigenfunctions

$$\psi(\alpha_i; f(z)) = C\sqrt{f'(z)}\tilde{\psi}(\tilde{\alpha}_i; z). \tag{6}$$

Once the desired eigenfunction is obtained in terms of the transformed variable, it may easily be expressed in terms of the true one by inverse transformation. Using (5) and comparing equations (2) and (4) term by term, we write

$$\tilde{V}(\tilde{\alpha}_i; z) - \tilde{E}(\tilde{\alpha}_i) = U(\alpha_i; z) \tag{7}$$

where

$$U(\alpha_i; z) \equiv f'^2\{V(\alpha_i; f(z)) - E(\alpha_i)\} + \frac{\hbar^2}{4m}\left\{\frac{3}{2}\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right\}. \tag{8}$$

The transformation function f has to be chosen such that the functional form of $U(\alpha_i; z)$ as given by (8) is identical to that of the known potential $\tilde{V}(\tilde{\alpha}_i; z)$. The energy eigenvalues $E(\alpha_i)$ can then be determined from the known values of $\tilde{E}(\alpha_i)$ and the parameters obtained through inverse mapping. Our scheme is in many ways similar to that proposed in [19].

To see how the method works, we consider the Rosen–Morse potential [5]

$$V(A, B, \alpha; x) = A^2 + \frac{B^2}{A^2} + 2B\tanh(\alpha x) - A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right)\operatorname{sech}^2(\alpha x). \tag{9}$$

Using the point canonical transformation

$$x \equiv f(z) = \frac{1}{\alpha}\tanh^{-1}(\cos(\alpha z)) \tag{10}$$

one obtains from equations (8) and (10)

$$U(A, B, \alpha; z) = \left[A^2 + \frac{B^2}{A^2} - \frac{\hbar^2\alpha^2}{8m} - E\right]\operatorname{cosec}^2(\alpha z) + 2B\operatorname{cosec}(\alpha z)\cot(\alpha z) - \left[A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right) + \frac{\hbar^2\alpha^2}{8m}\right]. \tag{11}$$

We now take the known potential to be the generalized Scarf potential [5]

$$\tilde{V}(\tilde{A}, \tilde{B}, \alpha; z) = -\tilde{A}^2 + \left(\tilde{A}^2 + \tilde{B}^2 - \frac{A\alpha\hbar}{\sqrt{2m}}\right) \operatorname{cosec}^2(\alpha z) - \tilde{B}\left(2\tilde{A} - \frac{\alpha\hbar}{\sqrt{2m}}\right) \operatorname{cosec}(\alpha z)\cot(\alpha z) \tag{12}$$

for which

$$\tilde{E}_n(\tilde{A}, \tilde{B}, \alpha) = \left(\tilde{A} + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 - \tilde{A}^2$$

$$\tilde{\psi}_n(\tilde{A}, \tilde{B}, \alpha; z) = [1 - \cos(\alpha z)]^{\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}-\tilde{B}}{2\alpha}\right)} [1 + \cos(\alpha z)]^{\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}+\tilde{B}}{2\alpha}\right)} \times P_n^{\left[\frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}-\tilde{B}}{\alpha}\right) - \frac{1}{2}, \frac{\sqrt{2m}}{\hbar}\left(\frac{\tilde{A}+\tilde{B}}{\alpha}\right) - \frac{1}{2}\right]}(\cos(\alpha z)) \tag{13}$$

are known. Using equations (12), (13) and (14a) in (7) and comparing like terms we get

$$A^2 + \frac{B^2}{A^2} - \frac{\hbar^2\alpha^2}{8m} - E = \tilde{A}^2 + \tilde{B}^2 - \frac{\tilde{A}\alpha\hbar}{\sqrt{2m}} \tag{14a}$$

$$B = -\tilde{B}\left(\tilde{A} - \frac{\alpha\hbar}{2\sqrt{2m}}\right) \tag{14b}$$

$$A\left(A + \frac{\alpha\hbar}{\sqrt{2m}}\right) + \frac{\hbar^2\alpha^2}{8m} = \tilde{A}^2 + \tilde{E} = \left(\tilde{A} + \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2. \tag{14c}$$

From equations (14b) and (14c) one obtains

$$A = \tilde{A} - \frac{\hbar\alpha}{2\sqrt{2m}} + \frac{n\alpha\hbar}{\sqrt{2m}}$$
$$B = -\left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)\tilde{B} = -\left(\tilde{A} - \frac{\hbar\alpha}{2\sqrt{2m}}\right)\tilde{B}. \tag{15}$$

Equation (14a) in conjunction with (15) gives the eigenvalues of the Rosen–Morse potential:

$$E_n(A, B, \alpha) = A^2 - \left(A - \frac{n\alpha\hbar}{\sqrt{2m}}\right)^2 + B^2\left(\frac{1}{A^2} - \frac{1}{(A - n\alpha\hbar/\sqrt{2m})^2}\right). \tag{16}$$

Also from equations (6), (13) and (15), the unnormalized eigenfunction is obtained after inverse transformation of the variable

$$\psi_n(A, B, \alpha; x) = [1 - \tanh(\alpha x)]^{p/2}[1 + \tanh(\alpha x)]^{q/2} P_n^{(p,q)}(\tanh(\alpha x))$$
$$\binom{p}{q} = \frac{\sqrt{2m}}{\hbar\alpha} A - n \pm \frac{\sqrt{2m}}{\hbar\alpha}\frac{B}{(A - n\alpha\hbar/\sqrt{2m})}. \tag{17}$$

The results given in equations (16) and (17) are the same as those obtained in [5] and [6] through operator techniques using the condition for shape invariance.

Similar mapping procedures can be followed starting from other types of shape invariant potentials. In table 1, we give the mapping functions for the Coulomb and Morse potentials which may be expressed in terms of the three-dimensional harmonic oscillator. (The one-dimensional harmonic oscillator is the $l=0$ special case of the three-dimensional oscillator, and its Hermite polynomial eigenfunctions are easily expressed as confluent hypergeometric functions [24].) In table 2, we present results for all other known types of shape invariant potentials like the Eckart, Poschl–Teller, etc, which can be mapped to the generalized Scarf potential. It is quite evident that the potentials in these two classes correspond to eigenfunctions which are represented by confluent hypergeometric and hypergeometric functions respectively. Finally, it should be mentioned that mappings via point canonical transformations can also be used to interrelate the reflection and transmission coefficients and S-matrices of various types of shape invariant potentials [25].

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